Girraween High School

Year 12 HSC Half Yearly Examination

MATHEMATICS EXTENSION 1

March 2015

Time Allowed: Two hours
(plus 5 minutes reading time)

Instructions To Students

- Attempt all questions.
- All necessary working must be shown for Questions 6 – 10.
- Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- For Questions 1 -5, write the letter corresponding to the correct answer on your answer sheet.
- For Questions 6 – 10, start each question on a new sheet of paper. Each question should be clearly labelled.

Section I

5 marks

Attempt Questions 1-5

Question 1 (1 mark)

The angle $70°$ in radians is:

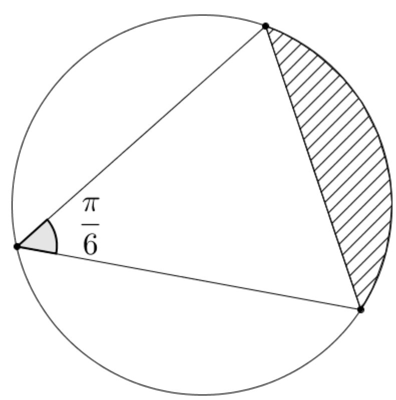
A. $\dfrac{7\pi}{18}$

B. $\dfrac{18\pi}{7}$

C. $\dfrac{7\pi}{36}$

D. $\dfrac{36\pi}{7}$

Question 2 (1 mark)



The diagram above is of a unit circle. The shaded area is given by:

A. $\dfrac{\pi}{12}$

B. $\dfrac{\pi}{6}$

C. $\dfrac{\pi - 3}{12}$

D. $\dfrac{2\pi - 3\sqrt{3}}{12}$

Question 3 on the next page

Question 3 (1 mark)

A polynomial of degree four is divided by a polynomial of degree two. What is the maximum possible degree of the remainder?

A. 0

B. 1

C. 2

D. 3

Question 4 (1 mark)

Which is the domain of $f(x) = \ln(x^2 - 1)$?

A. $x > 0$

B. $x > 1$

C. $x < -1$ and $x > 1$

D. $-1 < x < 1$

Question 5 (1 mark)

The number of different arrangements of the letters of the word REGISTER which begin and end with the letter R is:

A. $\dfrac{6!}{(2!)^2}$

B. $\dfrac{8!}{2!}$

C. $\dfrac{6!}{2!}$

D. $\dfrac{8!}{2!2!}$

Question 6 on the next page

Section II

86 marks

Attempt Questions 6-10

Write your answers on the paper provided.

In Questions 6-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (23 marks)

(a) Solve $\dfrac{x^2+5}{x} > 6$. [5]

(b) Let A be the point $(-2, 7)$ and let B be the point $(1, 5)$. Find the coordinates of the point P which divides the interval AB externally in the ratio $1 : 2$. [3]

(c) The graphs of the line $x - 2y + 3 = 0$ and the curve $y = x^3 + 1$ intersect at $(1, 2)$. Find the acute angle between the line and the tangent to the curve at the point of intersection. [5]

(d) The variable point $(3t, 2t^2)$ lies on a parabola. Find the Cartesian equation for this parabola. [2]

(e) i. Express $\sin x + \sqrt{3}\cos x$ in the form $A\sin(x + \alpha)$. [5]

ii. Hence, or otherwise, solve the equation $\sin x + \sqrt{3}\cos x = \sqrt{3}$ for $0 \leq x \leq 360°$. [3]

The exam continues on the next page

Question 7 (20 marks)

(a) Differentiate:

i. $y = \dfrac{1}{e^{x^2} + 1}$ [2]

ii. $y = \ln \dfrac{2}{x}$ [2]

iii. $y = \ln \sqrt{x(1-x)}$ [3]

(b) Consider the function $f(x) = -e^x + 1$.

i. State the domain and range of $f(x)$. [2]

ii. Prove that $f(x)$ is concave down for all x in the domain of $f(x)$, you must show working. [2]

(c) Find:

i. $\displaystyle\int_1^2 \frac{1}{5-2x}\,dx$ [2]

ii. $\displaystyle\int \frac{4e^{-x}}{1-2e^{-x}}\,dx$ [2]

(d) i. Solve $\log_2 x = \log_2 \dfrac{1}{x} + \log_2 (2x-1)$. [3]

ii. Solve $3^x = 18$. Give your answer to two decimal places. [2]

The exam continues on the next page

Question 8 (14 marks)

(a) i. Find the remainder obtained by dividing $P(x) = x^3 - bx^2 - bx + 4$ by $Q(x) = x - 2$. [1]

ii. Hence, or otherwise, find a value of the constant b such that $P(x)$ is divisible by $Q(x)$. [1]

iii. Find all the roots of $P(x)$ for this value of b. [3]

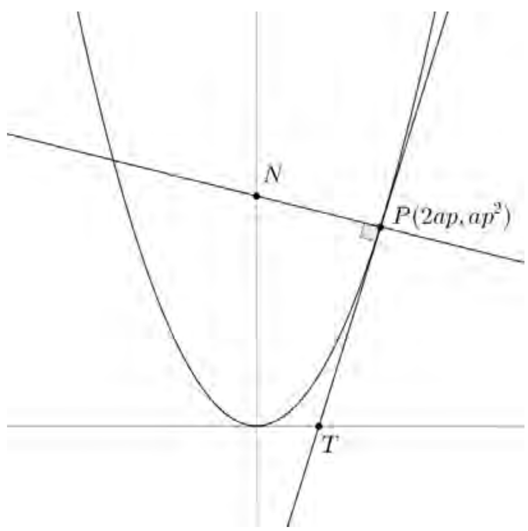
(b) Find the constant term in the expansion of $\left(3x^2 + \dfrac{5}{x^3}\right)^{10}$ [3]

(c) A particular exam contains 10 multiple choice questions, each with four choices. A student sitting this exam guesses all the answers randomly.

i. What is the probability that the students scores 50% in this exam? Give your answer to 3 decimal places. [2]

ii. What is the student's most likely score? [4]

Question 9 (19 marks)



(a) The diagram shows the graph of the parabola $x^2 = 4ay$. The tangent to the parabola at $P(2ap, ap^2)$ cuts the x-axis at T. The normal to the parabola at P cuts the y-axis at N.

i. Show that the equation of the tangent at P is $y = px - ap^2$ and find the coordinates of T. [3]

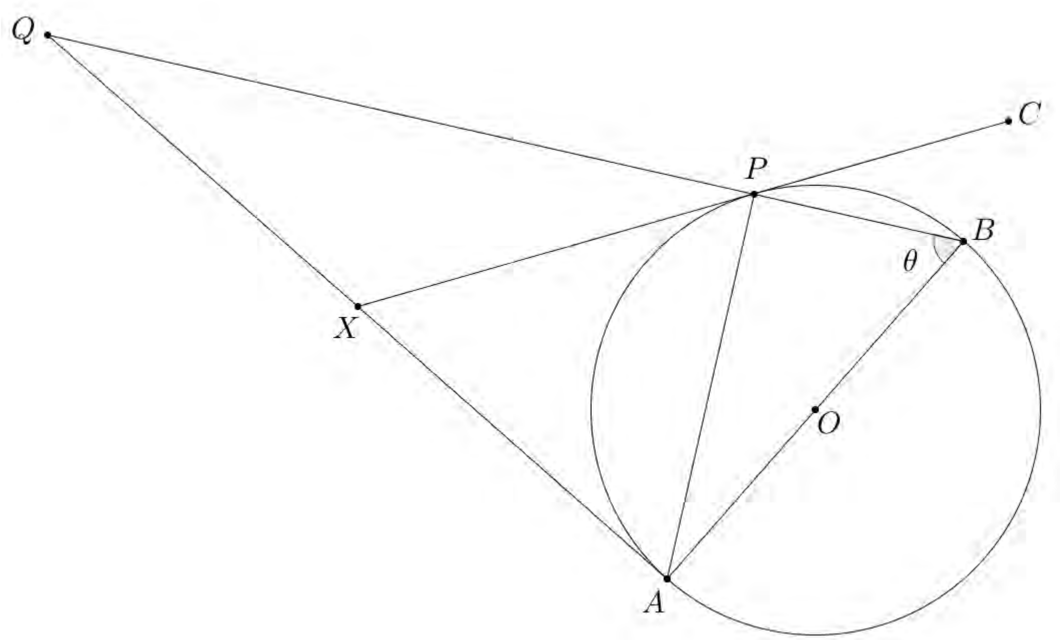
ii. Show that the coordinates of N are $(0, a(p^2 + 2))$. [2]

iii. Let M be the midpoint of NT. Show that the locus of M is a parabola and find its focal length. [4]

The exam continues on the next page

Question 9 (continued)

(b) In the diagram below AB is a diameter of the circle. The tangent AX and chord BP are produced to meet at Q. The tangent CP meets AQ at X.



i. If $\angle ABP = \theta$, show that $\angle XPQ = 90 - \theta$. [3]

ii. Show that X is the midpoint of AQ. [3]

(c) It can be shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all values of θ. (Do NOT prove this.) [4]

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin\theta$ for $0 \leq \theta \leq 2\pi$.

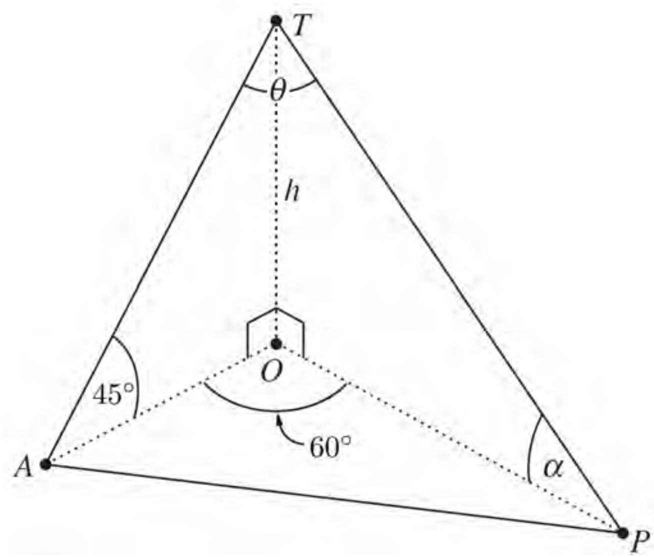
The exam continues on the next page

Question 10 (12 marks)

(a) Use mathematical induction to prove that, for integers $n \geq 1$, [4]

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

(b) The diagram below shows a vertical tower OT of height h. The angle of elevation of T from A is $45°$ and $\angle AOP = 60°$. Let the angle of elevation of T from P be α and let $\angle ATP = \theta$.



i. Show that $OA = h$ and $OP = h \cot \alpha$ [1]

ii. Show that [2]

$$AP^2 = h^2 + h^2 \cot^2 \alpha - h^2 \cot \alpha$$

iii. Using $\triangle ATP$, show that [2]

$$AP^2 = 3h^2 + h^2 \cot^2 \alpha - 2\sqrt{2}h^2 \operatorname{cosec} \alpha \cos \theta$$

iv. Show that [3]

$$\cos \theta = \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha$$

End of exam

Yr12 3U HSC TASK 2 SOLN

MC: A, D, B, C, C

Q1

$= 70 \times \frac{\pi}{180} = \frac{7\pi}{18} \quad \therefore$ A

Q2

$A = \frac{1}{2} r^2 (\theta - \sin\theta)$

but $\theta = \frac{\pi}{3}$ (∠ at centre is double ∠ at circumference).

$\therefore A = \frac{1}{2}\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}\left(\frac{2\pi - 3\sqrt{3}}{6}\right)$

$= \frac{2\pi - 3\sqrt{3}}{12} \quad \therefore$ D

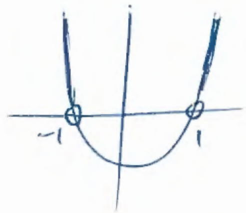
Q3

B

Q4

$x^2 - 1 > 0$

$(x-1)(x+1) > 0$



$\therefore$ D: $x < -1$ & $x > 1$ $\therefore$ C

Q5

REGISTER

$= \frac{6!}{2!} \quad \therefore$ C

Q6

(a) $\frac{x^2+5}{x} > 6$

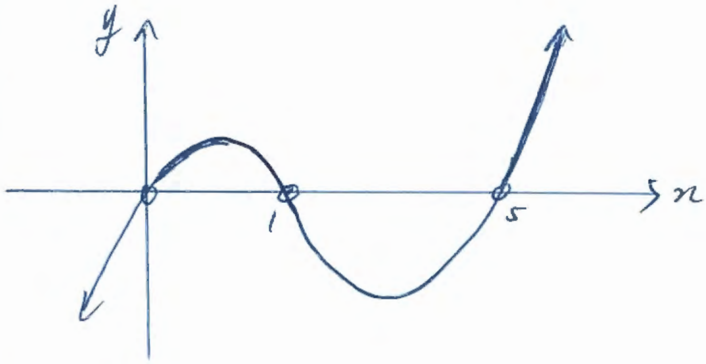
$x(x^2+5) > 6x^2$

$x(x^2+5) - 6x^2 > 0$

$x[(x^2+5) - 6x] > 0$

$x(x^2 - 6x + 5) > 0$

$x(x-5)(x-1) > 0$



$\therefore 0 < x < 1$ & $x > 5$

(b) $(-2, 7) \quad (1, 5)$

$-1 : 2$

$x = \frac{-1-4}{1} \qquad y = \frac{-5+14}{1}$

$= -5 \qquad = 9$

$\therefore P = (-5, 9)$

Q6/

(c)

$y = x^3 + 1$

$y' = 3x^2$

$y'(1) = 3$

$2y = x + 3$

$y = \frac{1}{2}x + \frac{3}{2}$

$m_1 = 3 \quad m_2 = \frac{1}{2}$

$\tan\theta = \left| \dfrac{3 - \frac{1}{2}}{1 + \frac{3}{2}} \right|$

$= \left| \dfrac{6-1}{2+3} \right| = |1| = 1$

$\therefore \theta = 45^0$

(d) $x = 3t$ & $y = 2t^2$

$t = \frac{x}{3}$

So $y = 2\left(\frac{x}{3}\right)^2 = \frac{2}{9}x^2$

$\therefore y = \frac{2}{9}x^2$

(e)

(i)

$\sin x + \sqrt{3}\cos x = A\sin(x+\alpha)$

$= A[\sin x \cos\alpha + \cos x \sin\alpha]$

$= A\cos\alpha \sin x + A\sin\alpha \cos x.$

$\therefore A\cos\alpha = 1$ & $A\sin\alpha = \sqrt{3}$

$A^2\cos^2\alpha + A^2\sin^2\alpha = 1 + 3 = 4.$

$\therefore A^2 = 4 \quad \therefore A = 2.$

$\therefore \cos\alpha = \frac{1}{2} \quad \therefore \alpha = 60°.$

$\therefore \sin x + \sqrt{3}\cos x = 2\sin(x+60)$

(ii)

$2\sin(x+60) = \sqrt{3}$

$\sin(x+60) = \frac{\sqrt{3}}{2}.$

$\therefore x + 60 = 60$ or 120

But $0 \le x \le 360$

$\therefore 60 \le x+60 \le 420$

$\therefore x + 60 = 60, 120, 420$

$\therefore x = 0°, 60°, 360°$

Q7

(a)

(i) $y = \dfrac{1}{e^{x^2}+1}$

$y = (e^{x^2}+1)^{-1}$

$y' = -(e^{x^2}+1)^{-2} \times (2xe^{x^2})$

$y' = \dfrac{-2xe^{x^2}}{(e^{x^2}+1)^2}$

(ii) $y = \ln\left(\frac{2}{x}\right)$

$y = \ln 2 - \ln x$

$y' = -\frac{1}{x}$

(iii) $y = \ln(x(1-x))^{\frac{1}{2}}$

$y = \frac{1}{2}\ln x(1-x)$

$y = \frac{1}{2}\left[\ln x + \ln(1-x)\right]$

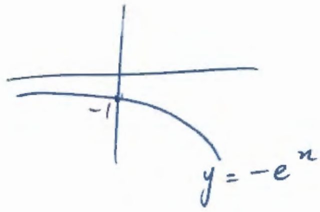
$y' = \frac{1}{2}\left[\frac{1}{x} - \frac{1}{1-x}\right]$

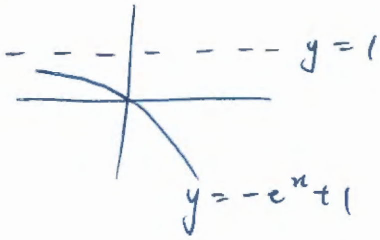
$y' = \frac{1}{2}\left[\frac{1-x-x}{x(1-x)}\right]$

$y' = \dfrac{1-2x}{2x(1-x)}$

(b)

(i) D: all real x





$\therefore$ R: $y < 1$

(ii) $y = -e^x + 1$

$y' = -e^x$

$y'' = -e^x$

Since $e^x > 0$ for all x

$\therefore -e^x < 0$ for all x

$\therefore f(x)$ is concave down for all x.

Q7

(c)

(i) $\int_1^2 \frac{1}{5-2x}\,dx$

$= -\frac{1}{2}\int_1^2 \frac{-2}{5-2x}\,dx$

$= -\frac{1}{2}\left[\ln(5-2x)\right]_1^2$

$= -\frac{1}{2}\left[\ln 1 - \ln 3\right]$

$= \frac{1}{2}\ln 3$

(ii) $\int \frac{4e^{-x}}{1-2e^{-x}}\,dx$

$= 2\int \frac{2e^{-x}}{1-2e^{-x}}\,dx$

$= 2\ln(1-2e^{-x}) + c$.

(d)

(i) $\log_2 x = \log_2 \frac{1}{x} + \log_2 (2x-1)$

$\log_2 x = -\log_2 x + \log_2(2x-1)$

$2\log_2 x = \log_2 (2x-1)$

$\log_2 x^2 = \log_2(2x-1)$

$\therefore\ x^2 = 2x-1$

$x^2 - 2x + 1 = 0$

$\therefore\ (x-1)^2 = 0 \quad \therefore\ x = 1$.

(ii) $3^x = 18$

$x = \log_3 18$

$x = \frac{\ln 18}{\ln 3} = 2.63$ (2dp)

Q8

(a)

(i) $r = P(2)$

$r = 8 - 4b - 2b + 4$

$r = 12 - 6b$

(ii) $b = 2$

(iii)

$P(x) = x^3 - 2x^2 - 2x + 4$

$$\begin{array}{r l}
 & x^2 - 2 \\
x-2 \,\big) & x^3 - 2x^2 - 2x + 4 \\
 & -(x^3 - 2x^2 \qquad\qquad) \\ \hline
 & \qquad\qquad -2x + 4 \\
 & \qquad\quad -(-2x + 4) \\ \hline
 & \qquad\qquad\qquad 0
\end{array}$$

$\therefore\ P(x) = (x-2)(x^2-2)$

$= (x-2)(x-\sqrt{2})(x+\sqrt{2})$

$\therefore$ roots are : $2,\ \pm\sqrt{2}$

Q8/

(b) $\left(3x^2+\frac{5}{x^3}\right)^{10}$

$T_{k+1} = \binom{10}{k}(3x^2)^{10-k}\left(\frac{5}{x^3}\right)^{k}.$

$= \binom{10}{k} 3^{10-k} x^{20-2k} 5^{k} x^{-3k}$

$= \binom{10}{k} 3^{10-k} 5^{k} x^{20-5k}$

For constant term : $20-5k=0$

$\therefore k=4.$

$\therefore$ constant term is :

$T_5 = \binom{10}{4} 3^6 5^4$

(c)

(i) Let X denote number of correct answers.

$P(X=5) = \binom{10}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^5$

$= 0.058$ (3dp)

(ii)

$$\frac{P_{k+1}}{P_k} = \frac{\binom{10}{k}\left(\frac{3}{4}\right)^{10-k}\left(\frac{1}{4}\right)^{k}}{\binom{10}{k-1}\left(\frac{3}{4}\right)^{11-k}\left(\frac{1}{4}\right)^{k-1}}$$

$$= \frac{10!}{(10-k)!\,k!} \times \frac{(11-k)!\,(k-1)!}{10!} \times \frac{4}{3} \times \frac{1}{4}$$

$$= \frac{11-k}{k} \times \frac{1}{3} = \frac{11-k}{3k}.$$

$\frac{11-k}{3k} \geq 1$ for greatest probability

$11-k \geq 3k$

$4k \leq 11$

$k \leq \frac{11}{4} = 2.75$

$\therefore k=2.$

$\therefore$ Most likely score is $2/10 = 20\%$.

Q9/

(a)

(i) $x^2 = 4ay$

So $y = \frac{1}{4a}x^2$

$y' = \frac{1}{2a}x$

$y'(2ap) = p$

$y - ap^2 = p(x-2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

$y=0$ when $px - ap^2 = 0$

$p(x-ap)=0$

$\therefore x = ap$

$\therefore T = (ap, 0)$

Q9

(a)

(ii) $m = -\frac{1}{p}$ pt $= (2ap, ap^2)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$x = 0$ when

$y - ap^2 = -\frac{1}{p}(-2ap)$

$y - ap^2 = 2a$

$y = 2a + ap^2 = a(2 + p^2)$

$\therefore N = (0, a(p^2+2))$

(iii)

$M = \left(\frac{ap+0}{2}, \frac{a(p^2+2)+0}{2}\right)$

$M = \left(\frac{1}{2}ap, \frac{1}{2}a(p^2+2)\right)$

$\therefore x = \frac{1}{2}ap \quad y = \frac{1}{2}a(p^2+2)$

So $p = \frac{2x}{a}$

So $y = \frac{1}{2}a\left(\frac{4x^2}{a^2} + 2\right)$

$y = \frac{2x^2}{a} + a$

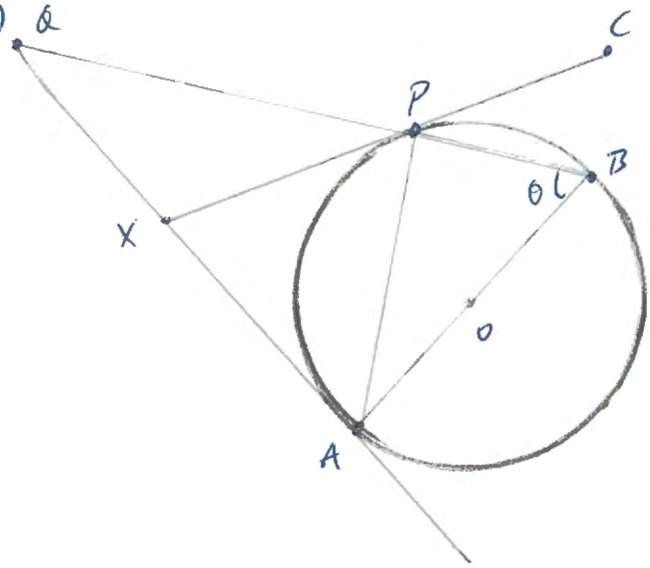
$y - a = \frac{2x^2}{a}$

$\frac{a}{2}(y - a) = x^2$

$4 \times (\text{focal length}) = \frac{a}{2}$ $\quad \therefore$ focal length is $\frac{a}{8}$

Q9

(b)



(i) $\angle APB = 90°$ ($\angle$ in semi-circle with diameter AB)

$\angle XPA = \theta$ ($\angle$ between tangent XP and chord AP equals $\angle$ in the alternate segment).

$\angle XPQ = 90 - \theta$ ($\angle$ sum of straight line QPB)

(ii)

$\angle QAB = 90°$ (tangent $\perp$ to radius at point of contact)

$\therefore \angle AQP = 90 - \theta$ ($\angle$ sum of ΔABQ)

$\therefore QX = XP$ (equal sides opposite equal $\angle$'s of ΔXQP)

Q9

(b)

(ii) $XP = XA$ (tangents from an external point)

$\therefore QX = XA$ $\therefore X$ is midpoint of QA.

(c)

$$\sin 3\theta + \sin 2\theta = \sin\theta$$

$$3\sin\theta - 4\sin^3\theta + 2\sin\theta\cos\theta = \sin\theta$$

$$4\sin^3\theta - 2\sin\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta\,[4\sin^2\theta - 2\cos\theta - 2] = 0$$

$$\sin\theta\,[4(1-\cos^2\theta) - 2\cos\theta - 2] = 0$$

$$\sin\theta\,[4 - 4\cos^2\theta - 2\cos\theta - 2] = 0$$

$$\sin\theta\,[-4\cos^2\theta - 2\cos\theta + 2] = 0$$

$$-\sin\theta\,[4\cos^2\theta + 2\cos\theta - 2] = 0$$

$$-\sin\theta\,(2\cos\theta - 1)(2\cos\theta + 2) = 0$$

$\therefore \sin\theta = 0$, $\cos\theta = \frac{1}{2}$, $\cos\theta = -1$

$\therefore \theta = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$

$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

Q10

(a)

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

Prove true for $n = 1$:

$LHS = \frac{1}{2!} = \frac{1}{2}$

$RHS = \frac{2! - 1}{2!} = \frac{1}{2!} = \frac{1}{2}$

$\therefore LHS = RHS$ $\therefore$ true for $n = 1$.

Assume true for $n = k$, i.e.

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k}{(k+1)!} = \frac{(k+1)! - 1}{(k+1)!}$$

Prove true for $n = k+1$:

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!}$$

$$LHS = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!}$$

$$= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!} = RHS.$$

$\therefore$ true for $n = k+1$

$\therefore$ By induction, it is true for all $n \geq 1$.

Q10

(b)

(i) In ΔATO:

$$\tan 45 = \frac{h}{OA}$$

$$\therefore OA = h$$

In ΔTOP:

$$\tan\alpha = \frac{h}{OP}$$

$$\therefore OP = h\cot\alpha$$

(ii) Using cosine rule on ΔAOP:

$$AP^2 = OA^2 + OP^2 - 2\times OA\times OP\times \cos 60$$

$$AP^2 = h^2 + h^2\cot^2\alpha - 2h^2\cot\alpha \times \frac{1}{2}$$

$$AP^2 = h^2 + h^2\cot^2\alpha - h^2\cot\alpha$$

(iii)

$$AT^2 = h^2 + h^2 = 2h^2$$

$$\therefore AT = \sqrt{2}h$$

$$TP^2 = h^2 + h^2\cot^2\alpha$$

$$= h^2(1+\cot^2\alpha)$$

$$\therefore TP = h\sqrt{1+\cot^2\alpha}$$

$$= h\operatorname{cosec}\alpha$$

(iii)

So:

$$AP^2 = AT^2 + TP^2 - 2\times AT\times TP\times\cos\theta$$

$$= 2h^2 + h^2 + h^2\cot^2\alpha - 2\sqrt{2}h^2\operatorname{cosec}\alpha\cos\theta$$

$$= 3h^2 + h^2\cot^2\alpha - 2\sqrt{2}h^2\operatorname{cosec}\alpha\cos\theta$$

(iv)

Equating the expressions for AP^2 from part (ii) & (iii) we get:

$$h^2 + h^2\cot^2\alpha - h^2\cot\alpha =$$
$$3h^2 + h^2\cot^2\alpha - 2\sqrt{2}h^2\operatorname{cosec}\alpha\cos\theta$$

$$2h^2 + h^2\cot\alpha - 2\sqrt{2}h^2\operatorname{cosec}\alpha\cos\theta = 0$$

$$2 + \cot\alpha - 2\sqrt{2}\operatorname{cosec}\alpha\cos\theta = 0$$

$$2 + \frac{\cos\alpha}{\sin\alpha} - 2\sqrt{2}\frac{\cos\theta}{\sin\alpha} = 0$$

$$2\sin\alpha + \cos\alpha - 2\sqrt{2}\cos\theta = 0$$

$$2\sqrt{2}\cos\theta = 2\sin\alpha + \cos\alpha$$

$$\cos\theta = \frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\cos\alpha$$